On the Cardioids fulfilling Certain Assigned Conditions

By

SISTER MARY GERVASE, M.A.

of

THE SISTERS OF CHARITY, HALIFAX, N. S.

A DISSERTATION

Submitted to the Catholic Sisters College of the Catholic University of America in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy

WASHINGTON, D. C.
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Introductory

The tricuspidal, bicircular quartic of the third class defined by the

$$\left.\begin{array}{l}\text{Cartesian equation } (x^2+y^2+ax)^2=a^2(x^2+y^2)\\ \text{polar equation } \rho=a(1-\cos\theta)\end{array}\right\},$$

and commonly known as the Cardioid, has for many years been the object of mathematical investigation. It has lately been studied by Raymond Clare Archibald in his Inaugural-Dissertation "The Cardioide and Some of Its Related Curves" (Strassburg, 1900), which work contains an historical sketch of the curve and a presentation of results prior to the year of its publication. Since then, the only work on the subject of considerable length is Professor Archibald's paper, "The Cardioid and Tricuspid: Quartics with Three Cusps."* Besides this there have appeared a few detached problems† and contributions in periodicals treating the curve from either a metric or a projective standpoint.

The chief characteristic of former research along this line seems to be the examination of the cardioid as a fixed curve and the consideration of its properties as such. The present investigation starts from a different point of view, which we may outline as follows:

In general, a curve of the fourth degree is capable of satisfying 14 conditions. The cardioid, however, having 3 cusps, two of which are at the fixed (circular) points I and J, can be subjected to only 4 conditions.‡ If, then, 3 conditions be imposed, there are ∞^1 curves satisfying them; therefore, the special elements (cusp, focus, double tangent) describe definite loci. If 4 conditions are given, there are a finite number of curves satisfying them. It is our purpose to obtain the loci generated in the first case; and, in the second, to determine the number and (where possible) the reality of cardioids for various kinds of assigned conditions.

The co-ordinate system which lends itself most readily to such an investigation is the system of conjugate (also called circular) co-ordinates, in which a point is named by a vector which has its

* *Annals of Mathematics*, 2d S. 4, 1902-3, pp. 95ff.

† Cf., for example, Questions 14,435; 16,266; 16,392. *Ed. Times* (London).

‡ Cusps at I and J count for 4 conditions each. The third cusp counts for 2 more conditions. Thus, the 14 conditions reduce to $14-(2\times4+2)=4$.

initial extremity at some fixed point, called the origin. Denoting a vector by z, its projections on the real and the imaginary axes are designated by x, y, respectively; and

$$z=x+iy.*$$

With a vector z is associated its conjugate z'. This may be geometrically defined as its reflection in the real axis. Accordingly,

$$z'=x-iy.$$

All vectors may be considered as obtained from the standard unit vector by means of a stretch and a turn. Turns are regularly designated by the letter t; fixed turns, by $t_1, t_2, t_3 \ . \ . \ .$; variable turns, by $t, t^1, \tau \ . \ . \ .$†

Regarding the cardioid as the epicycloid generated by equal circles, the map-equation of the curve is

$$z=r(2t-t^2).$$

The centre of the fixed circle is at O, which point, we shall, with Professor Morley,‡ call the centre of the cardioid. This point is also the singular focus of the curve. The cusp is at $z=r$ (r being on the real axis).

This map-equation involves its conjugate,

$$z'=r\left(\frac{2}{t}-\frac{1}{t^2}\right).$$

These two equations taken together may be considered the parametric equations of the curve.

A cardioid with centre at z_0 and any orientation has for its map-equation:

$$z-z_0=2at-a't^2$$

$$z=z_0+\frac{a^2}{a'} \text{ gives the cusp.}$$

The angle made by the axis of the curve with the axis of reals is θ, where

$$e^{2i\theta}=\left(\frac{a}{a'}\right)^3.$$

---

* The symbol i denotes, as usual, $\sqrt{-1}$.

† Cf. Morley, "On Reflexive Geometry," *Transactions of the American Mathematical Society*, 1907, Vol. 8, p. 14.

‡ "Metric Geometry of the Plane N-Line," *Transactions of the American Mathematical Society*, 1900, Vol. 1, p. 105.

Any tangent is given by

$$(z-z_0)-(z'-z'_0)t^3-3at+3a't^2=0;$$

the double tangent, by

$$a'^3(z-z_0)+a^3(z'-z'_0)=3a^2a'^2.$$

The kinds of data we shall consider are: point, line, centre, cusp, or double tangent given, to solve a specified problem. The last three are equivalent to 2 conditions each. The problems arising naturally fall into 2 classes:

I. Given 3 conditions, find specified loci; as,

(1) Given the centre and a line; find the locus of cusps.
(2) Given the centre and a point; find the locus of cusps.
(3) Given the cusp and a line; find the locus of centres.
(4) Given the cusp and a point; find the locus of centres.
(5) Given the double tangent and a line; find the locus of cusps.
Given the double tangent and a line; find the locus of centres.
(6) Given the double tangent and a point; find the locus of cusps.
Given the double tangent and a point; find the locus of centres.
(7) Given 3 lines, find the locus of centres.
(8) Given 2 lines and 1 point, find the locus of centres.
(9) Given 1 line and 2 points, find the locus of centres.
(10) Given 3 points, find the locus of cusps.

II. Given 4 conditions, find how many solutions there are; as,

(a) Given the centre and 2 lines, how many cardioids are there? How many are real?
(b) Given the centre, a point and a line. Apply the same questions.
(c) Given the centre and 2 points.
(d) Given the cusp and 2 lines.
(e) Given the cusp, 1 line and 1 point.
(f) Given the cusp and 2 points.
(g) Given the double tangent and 2 lines.
(h) Given the double tangent, 1 line and 1 point.
(i) Given the double tangent and 2 points.
(j) Given 4 lines.
(k) Given 3 lines and 1 point.
(l) Given 2 lines and 2 points.

(m) Given 1 line and 3 points.
(n) Given 4 points.

Again, there is an evident division of the problems into simpler and more difficult cases. It is not the intrinsic value of some of the simple cases which authorizes their appearance in this work, but rather their importance in the solution of some of the more difficult problems.

Before taking up the problems, it will be well to indicate a guiding principle which will be found of great importance for cases where the cardioids are to touch several lines. It may be stated thus:

If θ be the angle between 2 lines which touch a cardioid, the angle between the cuspidal rays to the points of tangency has some one of the values $\frac{2}{3}\theta$, $\frac{2}{3}(\theta+360°)$, $\frac{2}{3}(\theta+720°)$.

For the case of parallel tangents, this specializes to the well-known theorem:

The points of contact of any 3 parallel tangents to a cardioid subtend angles of 120° at the cusp.

With these considerations premised, we shall proceed to a treatment of the simpler cases.

The Simpler Cases

I. Given 3 Conditions.

PROBLEM (1): Given the center and a line; find the locus of cusps.

Let us take the centre at O; the line, as $z+z'=2$.

The map-equation of the cardioid is $z=2at-a't^2$

The cusp: $z=\frac{a^2}{a'}$

Any tangent: $z-3at+3a't^2-z't^3=0$

If the line $z+z'=2$ is to touch the cardioid, the following identity must exist:

$$z-3at+3a't^2-z't^3 \equiv z+z'-2:$$

whence, $$t^3=-1 \qquad (1)$$

and $$-3at+3a't^2=-2 \qquad (2)$$

From (2) we get $t=-1$, $-\omega$, or $-\omega^2$*

It will be sufficient to take one of these values; for, since the expression for the cusp $\left(\frac{a^2}{a'}\right)$ does not occur rationally in (2), the rationalized form of the equation will be the same, no matter which value is selected for t. Cubing (2) and substituting the value $t=-1$, we obtain

$$27(a^3+a'^3)-54aa'+8=0.$$

Now the cusp is given by $z=\frac{a^2}{a'}$, and we seek the cusp locus. Therefore, if we call $\frac{a^2}{a'}$, k [whence result the equations:

$$\frac{a'^2}{a}=k'$$
$$aa'=kk'$$
$$a^3+a'^3=kk'(k+k')],$$

the equation of the locus becomes

$$27kk'(k+k')-54kk'+8=0.$$

This is a rational cubic with its double point at $k=\frac{2}{3}$ and vertex

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*ω and ω^2 are the imaginary cube roots of unity.

at $k=-\frac{1}{3}$ (Fig. I). The line $z+z'=2$ is an asymptote to the curve.

Making the equation homogeneous, it becomes

$$27kk'(k+k'-2w)+8w^3=0,$$

which is of the form

$$\alpha\varphi+\beta^3\psi=0^*$$

In this form it is easily seen that the curve has points of inflection at the intersections of $k=0$ and $w=0$, and of $k'=0$ and $w=0$,

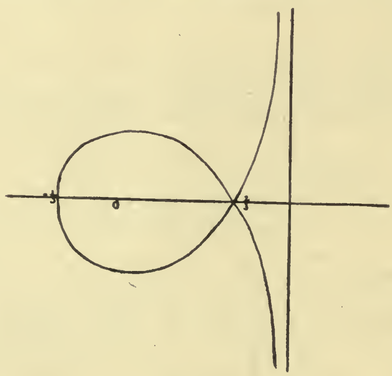

FIG. I.

respectively; i. e., at the circular points, I and J. Moreover, $k=0$ is the flex tangent at I; $k'=0$, the flex tangent at J.

If the given line be taken to be

$$z=z't_1+a_1,$$

the cusp locus becomes the circular cubic

$$27t_1kk'(k-k't_1-a_1)-a_1^3=0,$$

which also has points of inflection at I and J.

PROBLEM (2): Given the centre and a point; find the locus of cusps.

Let us take the centre at 0; the point, as $z=1$.

Map-equation of the cardioid: $z=2at-a't^2$.

Since the cardioid is to be on the point $z=1$, we have the equation

$$2at-a't^2=1$$

* Cf. Salmon, *Analytische Geometrie der höheren ebenen Kurven*, Leipzig, 1882 (Zweite Auflage), p. 50.

which involves its conjugate

$$2\frac{a'}{t}-\frac{a}{t^2}=1.$$

From these two equations it is necessary to eliminate t and express the result in terms of $\frac{a^2}{a'}$, which, as before, we shall designate as k.

The elimination of t gives

$$3a^2a'^2+6aa'-4(a^3+a'^3)=1.$$

In terms of k, k' the locus is

$$3k^2k'^2+6kk'-4kk'(k+k')=1.$$

This bicircular quartic (see Fig. IV, p. 20) has its vertex at $-\frac{1}{3}$ and cusp at 1. Expressed in homogeneous form

$$kk'(3kk'-4kw-4k'w+6w^2)-w^4=0,$$

it is easy to see that the curve intersects $k=0$, $k'=0$ in 4 coincident points. The form

$$k'^2k(3k-4w)+w(6kk'w-4k^2k'-w^3)=0$$

shows that the curve goes through I, the tangents thereat being $k=0$ and $3k-4w=0$. But since there are 4 coincident points at I, $k=0$ must be a flex tangent; that is, I is a flecnode. Similarly, the tangents at J are $k'=0$ and $3k'-4w=0$, the former being a flex tangent.

When p is the given point, the cusp locus is the bicircular quartic

$$3k^2k'^2-4kk'(kp'+k'p)+6pp'kk'-p^2p'^2=0,$$

which has flecnodal points at I and J with $k=0$ and $k'=0$ as flex tangents.

PROBLEM (3): Given the cusp and a line; find the locus of centres.

Let the cusp be at 0; the line, $z+z'=2$.

The equation of a cardioid with cusp at 0 is

$$z+\frac{a^2}{a'}=2at-a't^2;$$

i. e., $$z=-\frac{(a-a't)^2}{a'}$$

The centre is given by $z=-\frac{a^2}{a'}$,

Any tangent to this is

$$z-z't^3-3at+3a't^2-\frac{a'^2t^3}{a}+\frac{a^2}{a'}=0.$$

If $z+z'=2$ is to be tangent,

$$z-z't^3-3at+3a't^2-\frac{a'^2t^3}{a}+\frac{a^2}{a'}\equiv z+z'-2.$$

From this identity, it follows that

$$t^3=-1 \qquad\qquad (1)$$

and

$$-3at+3a't^2-\frac{a'^2t^3}{a}+\frac{a^2}{a'}=-2 \qquad\qquad (2)$$

From (1), $t=-1$, $-\omega$, or $-\omega^2$.

Cubing (2) and substituting the value $t=-1$, we obtain

$$27[a^3+a'^3+3aa'(a+a')]=-\frac{a^2}{a'}-\frac{a'^2}{a}-2.$$

Calling $-\frac{a^2}{a'}$, c; $-\frac{a'^2}{a}$, c', the equation of the locus reduces to

$$(c+c'-2)^3+54cc'=0.$$

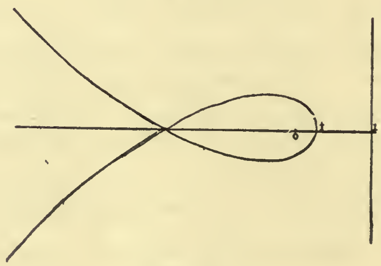

FIG. II.

This is a nodal cubic, known as the Tschirnhausen Cubic,* with vertex at $\frac{1}{4}$ and double point at -2 (Fig. II).

If the given line is $z=z't_1+a_1$, the centre locus becomes the nodal cubic $(c-c't_1-a_1)^3-27cc'a_1t_1=0$.

PROBLEM (4): Given the cusp and a point; find the locus of centres.

Let the cusp be at 0; the point, at $z=1$.

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* It is also called the Cubique de l'Hopital. Cf. Archibald, *op. cit.*, p, 19, where he states that "the locus of the vertices of co-cuspidal cardioides tangent to a given line is a Tschirnhausen Cubic."

The equation of a cardioid with cusp at 0 is

$$z=-\frac{(a-a't)^2}{a'} \qquad \left(\text{centre: } c=-\frac{a^2}{a'}\right)$$

The conditions that $z=1$ be on the cardioid are expressed by

$$\left.\begin{aligned} 2at-a't^2-\frac{a^2}{a'} &= 1 \\ 2\frac{a'}{t}-\frac{a}{t^2}-\frac{a'^2}{a} &= 1 \end{aligned}\right\}.$$

Eliminating t from these equations and expressing the resulting equation in terms of c and c', we obtain as the sought centre locus:

$$4cc'=(c+c'-1)^2$$

This is a parabola with vertex at $\frac{1}{4}$ and focus at 0.

When p is the given point, the locus is the parabola confocal with the preceding:

$$4cc'pp'=(cp'+c'p-pp')^2.$$

PROBLEM (5): Given the double tangent and a line; find the locus of centres.

Let the given line be $z=t_1z'$; the double tangent, $z+z'=2$. The equation of the cardioid may be taken to be

$$z-z_o=r(2t-t^2), \text{ where } r \text{ is real.}$$

That the given conditions be fulfilled, the following identities must exist:

$$(z-z_o)-(z'-z'_o)t^3-3rt(1-t)\equiv z-z't_1 \dots\dots\dots (1)$$
$$(z-z_o)+(z'-z'_o)-3r\equiv z+z'-2 \dots\dots\dots (2)$$

From (1) $t^3=t_1$ [whence, $t=\tau$, $\omega\tau$, $\omega^2\tau$, where $\tau=\sqrt[3]{t_1}$]

and $$z_o+z'_o+3r=2.$$

Combining these conditions and eliminating r and t, we find the locus breaks up into the 3 lines given by the equations:

$$z_o=z'_o\,\tau-\frac{2\tau(1-\tau)}{1-\tau+\tau^2}$$

$$z_o=z'_o\,\omega\tau+\frac{2\omega\tau(\omega\tau-1)}{1+\omega^2\tau^2-\omega\tau}$$

$$z_o=z'_o\,\omega^2\tau+\frac{2\omega\tau(\tau-\omega)}{1+\omega\tau^2-\omega^2\tau}$$

These three lines pass through the point of intersection of $z=z't_1$

and $z+z'=2$. If θ be the inclination of the given line, $z=z't_1$, the first line of the centre locus is inclined to the axis of reals at an angle equal to $\frac{\theta}{3}$; the others at angles $\frac{\theta+\pi}{3}$, $\frac{\theta+2\pi}{3}$, respectively.

The locus of cusps of cardioids which have $z+z'=2$ as double tangent and touch the line $z-z't_1=0$ consists of 3 lines through the intersection of $z+z'=2$ and $z-z't_1=0$ such that if α be the angle the centre locus makes with the double tangent, and β be the angle the cusp locus makes with the same tangent, then

$$\beta=\tan^{-1}\left(\frac{1}{3}\tan\alpha\right).$$

PROBLEM (6): Given the double tangent and a point; find the locus of cusps.

Let us take the point at the origin, and the equation of the cardioid as:

$$\begin{aligned} z&=z_o+r-r+r(2t-t^2), \qquad \text{where } r \text{ is real,} \\ &=k-r(1-t)^2 \end{aligned}$$

For the fulfillment of the given conditions, the following identities must exist:

$$(z-k)+(z'-k')-r\equiv z+z'-2 \quad\ldots\ldots\ldots\ldots(1)$$

$$k-r(1-t)^2=0 \quad\ldots\ldots\ldots\ldots(2)$$

$$k'-r\left(1-\frac{1}{t}\right)^2=0 \quad\ldots\ldots\ldots\ldots(3)$$

From (1) $$k+k'+r=2 \quad\ldots\ldots\ldots\ldots(4)$$

Eliminating t and r from (2), (3), (4), we obtain as the required locus:

$$4(2-k-k')^2kk'=[(-k-k')(2-k-k')+kk']^2$$

As an aid in the discussion of the curve, let us transform it to Cartesian co-ordinates by means of the equations

$$\left.\begin{aligned} k&=x+iy \\ k'&=x-iy \end{aligned}\right\}.$$

The equation becomes

$$9x^4-6x^2y^2+y^4+24xy^2-8x^3-16y^2=0$$

This is of the form

$$y^2\varphi+x^3(9x-8)=0,$$

which indicates a cusp at the origin with $y=0$ as the cusp tangent; the curve has also a branch cutting the X-axis at the point $\left(\frac{8}{9}, 0\right)$.

Noting the behavior of the curve at infinity, we observe that it has 2 parabolic branches with directions determined by $y = \pm\sqrt{3}x$; i. e., the infinite branches tend towards angles of $\pm 120°$ with the X-axis.

Where p is the given point, the resulting locus takes the form:

$$4(2-k-k')^2(k'-p')(k-p) = [(2-k-k')(p'-k'+p-k) + (p'-k')(p-k)]^2,$$

which has one cusp at p and 2 parabolic branches with directions determined by $\frac{k}{k'} = \frac{1 \pm \sqrt{3}i}{2}$.

The locus of centres of cardioids which have $z+z'=2$ for double tangent and pass through the origin is the quartic:

$$3x^4 - 18x^2y^2 + 27y^4 + 16x^3 + 144xy^2 + 24x^2 - 72y^2 - 16 = 0.$$

This curve has 2 parabolic branches tending towards angles of $\pm 150°$ with the X-axis. From the form

$$y^2(27y^2 - 18x^2 + 144x - 72) + (x+2)^3(3x-2) = 0,$$

it is easily seen that the curve has a cusp at the point $(-2, 0)$ with $y=0$ as the cusp tangent, as well as a branch cutting the X-axis at the point $\left(\frac{2}{3}, 0\right)$.

The foregoing complete the simpler cases when 3 conditions are given. Let us now proceed to the cases arising from them.

II. Given 4 Conditions.

In these problems we shall first endeavor to find the number of cardioids which fulfil 4 conditions as variously assigned. In the cases here treated this number is found by the intersections of the various loci already obtained. Such a solution is, however, the maximum number and includes the imaginary curves, if there are any. It will therefore be our next problem to separate these imaginary solutions and thus determine the number of real cardioids for each case.

PROBLEM (a): Given the centre and 2 lines.

Let the given centre be at 0; the 2 lines, $z+z'=2$ and $z=z't_1+a_1$. We seek the number of cardioids which have 0 for centre and touch each of the two given lines.

The cardioids which have 0 for centre and touch the line $z+z'=2$ have their cusps on the circular cubic

$$27kk'(k+k') - 54kk' + 8 = 0.$$

Similarly, the cardioids having 0 for centre and touching the line $z=z't_1+a_1$ have their cusps on the circular cubic

$$27t_1kk'(k-k't_1-a_1)-a_1^3=0.$$

Thus, the cusps of cardioids which have 0 for centre and touch both of the given lines are given by the points of intersection of these two circular loci. These cubics have 9 intersections. Therefore, there must be 9 cusps of cardioids, and hence (since cusp and centre uniquely determine a cardioid) 9 cardioids, which fulfill the required conditions. But, since the loci have points of inflection at the circular points and the same tangents thereat, 3 of their intersections are at each of these imaginary points. Thus, 6 of the 9 cardioids are imaginary. The other 3 are always real. For, considering the two given lines, either

I. They will be equidistant from the given centre; or,

II. One will lie nearer that point than the other. Now the circle with radius equal to the minimum radius vector of either cubic lies wholly within the loop of that cubic. The vertex of the second cubic will, then, lie either on the circumference of this circle (Case I) or within it (Case II). In either case, this point lies within the loop of the first cubic. The second curve, in tending towards its asymptote, must intersect the loop of the first in 2 real points, and only in 2. But if there are but 3 possible real intersections, and two have been shown real, the third must necessarily be real. Thus, there are 3 real cardioids which have a given centre and touch 2 given lines. (See Fig. X, p. 44.)

PROBLEM (b): The centre, a point and a line given.

Take the origin as centre; $z+z'=2$ as the given line and p as the given point.

The intersections of cusp-loci will, as before, give the number of solutions. The cusp-locus for cardioids with centre 0 and touching $z+z'=2$ is the circular cubic

$$27kk'(k+k')-54kk'+8=0;$$

the cusp-locus for cardioids with centre 0 and passing through p is the bicircular quartic

$$3k^2k'^2-6pp'kk'-4kk'(kp'+k'p)-p^2p'^2=0.$$

These two curves have 12 common intersections; i. e., there should be 12 cardioids satisfying the required conditions. But, since

the two curves have 4 points in common at each of the circular points, there can be, at most, 4 real cardioids. However, there are not always 4 real. Therefore, let us determine the regions of the plane where there are 4, or only 2, or no real cardioids satisfying the given conditions.

The map-equation of any cardioid with centre at 0 is

$$z = 2at - a't^2$$

Since $z + z' = 2$ is to be tangent to this,

$$z - 3at + 3a't^2 - z't^3 \equiv z + z' - 2$$

Employing the value $t = -1$, we have for a, a', any one of the 3 equations:

$$\begin{aligned} 3a + 3a' &= -2 \\ 3a\omega + 3a'\omega^2 &= -2 \\ 3a\omega^2 + 3a'\omega &= -2 \end{aligned}$$

It will be sufficient to take the first equation. Since p is the given point, we have

$$\left.\begin{aligned} 2at - a't^2 &= p \\ 2\frac{a'}{t} - \frac{a}{t^2} &= p' \end{aligned}\right\}$$

whence, eliminating a, a', we have for t the quartic

$$\begin{vmatrix} 3 & 3 & -2 \\ 2t & -t^2 & p \\ -\frac{1}{t^2} & \frac{2}{t} & p' \end{vmatrix} = 0;$$

i. e., $$p't^4 + 2p't^3 + 2t^2 + 2pt + p = 0.$$

This shows, as before determined, that there are at most 4 cardioids satisfying the specified conditions. We shall now seek the regions of the plane where there are 4 real, 2 real, or none.

Now, for a quartic $(a, b, c, d, e)(x, 1)^4$ with realc oefficients,
$\Delta = 0$ indicates the coincidence of 2 of the 4 roots;
$\Delta < 0$ gives 2 real and 2 imaginary roots;
$\Delta > 0$ gives 0 or 4 real roots;
$\Delta > 0$ and, besides, $b^2 - ac > 0$ and $12(b^2 - ac)^2 - (ae - 4bd + 3c^2)a^2 > 0$ gives 4 real roots.*

The locus $\Delta = 0$ will, then, separate the different regions of the

* Cf. Halphen, *Traité des Fonctions Elliptiques et Leurs Applications*, Paris, 1886, Première Partie, p. 123.

plane which we seek. In order to apply these criteria, transform the unit circle into the real axis by the transformation

$$t=-\frac{x-i}{x+i},$$

by which the quartic becomes

$$x^4(2-s_1)+2x^2(2-3s_1)+8ix(p'-p)+3s_1+2=0.*$$

The discriminant of this quartic is

$$\Delta=-256pp'(p+p'-2)[27pp'(p+p'-2)+8]$$

$$b^2-ac=-\frac{1}{3}(2-s_1)(2-3s_1)$$

$$12H^2-Ia^2=\frac{4}{3}(2-s_1)^2\,[(2-3s_1)^2-4]$$

The discriminant, equated to 0 and geometrically interpreted, breaks up into the lines $0I$, $0J$ (which, being imaginary, may be disregarded), the line $p+p'-2=0$ and the circular cubic $27pp'(p+p'-2)+8=0$. Plotting these, we can easily mark the sought regions of the plane.

For points on the cubic and its asymptote, $\Delta=0$; i. e., there are 2 of the 4 cardioids coincident. Note that p given in such a position would be a cusp (on the cubic) or a point of tangency (on the line). For points not on these curves, Δ is either >0 or <0; hence, these curves mark off the regions of the plane where there are 4, 2, or 0 real curves.

Let us test points in the various sections. For point $\frac{1}{2}$, $\Delta>0$ and $12H^2-Ia^2<0$. Therefore, for points in the region where $\frac{1}{2}$ is (i. e., in the loop), there are no real cardioids. This was to be expected, for 0 is the given centre and the cubic is the cusp-locus for cardioids touching $z+z'=2$. Therefore, p taken between the centre and the cusp would certainly lead to no real cardioid.

For point 2, $\Delta<0$, which indicates that p in the region exterior to the circular cubic and its asymptote gives 2 real and 2 imaginary cardioids.

For point $\frac{5}{6}$, $\Delta>0$, $b^2-ac>0$ and $12H^2-Ia^2>0$; therefore, in the region between the cubic and its asymptote, p gives 4 real cardioids.

$^*s_1=p+p'$.

Thus, p will determine the number of real cardioids according to its position in the various regions marked in Fig. III.

It is worthy of remark that, taking 0 as the given point, $z+z'=2$ as the given line and c as the centre, the discriminant of the resulting quartic in t is the parabolic cubic

$$54cc'+(c+c'-2)^3=0,$$

which forms the centre-locus for cardioids with cusp 0 and line $z+z'=2$. Centres in the loop give imaginary solutions; between the infinite branches there are 4 real cardioids; centres at other points in the plane yield 2 real and 2 imaginary curves.

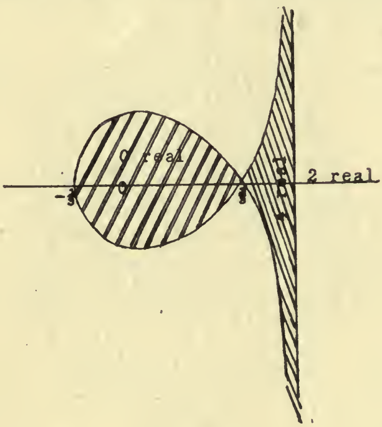

Fig. III.

PROBLEM (c): Given the centre and 2 points.

Take the centre at the origin; the points as 1 and p.

The cusp-locus for cardioids with centre 0 and passing through 1 is

$$3k^2k'^2+6kk'-4kk'(k+k')-1=0.$$

The cusp-locus for cardioids with centre 0 and passing through p is

$$3k^2k'^2+6kk'pp'-4kk'(kp'+k'p)-p^2p'^2=0.$$

These two curves have 16 common intersections; i. e., there should be 16 cardioids satisfying the given conditions. But, since the two curves have double points at I and J with one branch of each having a flex thereat, and the flex tangents also being common, there are 12 intersections at I and J. Therefore, there can be, at most, 4 real cardioids. Let us now determine the regions of the

plane where there are 4, 2, or 0 curves satisfying the required conditions.

From the conditions that the sought curves have centre 0 and pass through 1 and p, we obtain the equations:

$$\left.\begin{aligned}2at-a't^2&=p\\ 2\frac{a'}{t}-\frac{a}{t^2}&=p'\end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned}2a\tau-a'\tau^2&=1\\ 2\frac{a'}{\tau}-\frac{a}{\tau^2}&=1\end{aligned}\right\}.$$

From these a, a', τ must be eliminated. The elimination of these quantities leads to the quartic in t^3:

$$\begin{aligned}4p'^3(p'-1)t^{12}&+4(8p'^3-5pp'^3+6pp'^2-9p'^2)t^9+3(11p^2p'^2-16p^2p'\\&-16pp'^2+30pp'-9)t^6+4(8p^3-5p'p^3+6p^2p'-9p^2)t^3\\&+4p^3(p-1)=0.\end{aligned}$$

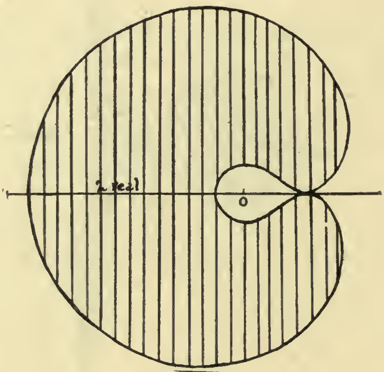

FIG. IV.

For this the invariants are

$$4I=3^3(s_2-1)(s_2^3+21s_2^2-32s_1s_2+51s_2-9)$$

$$\begin{aligned}8J=3^3(&-s_2^6+66s_2^5-80s_2^4s_1-3s_2^4+48s_2^3s_1+108s_2^3+585s_2^2\\&-624s_1s_2^2+128s_1^2s_2^2+144s_1s_2-270s_2+27)\end{aligned}$$

$$\begin{aligned}\Delta=3^9s_2^3(4s_1+s_2^2-6s_2-3)&\left\{\begin{aligned}&-2^6s_2s_1^3+2^4s_1^2(-2s_2^3+15s_2^2\\&\qquad+12s_2-1)\\&+2^2s_1(-s_2^5+18s_2^4-54s_2^3-128s_2^2\\&\qquad-33s_2+6)\\&+(3s_2^6-30s_2^5+17s_2^4+300s_2^3+213s_2^2\\&\qquad+18s_2-9)\end{aligned}\right.\\&=3^9s_2^3(4s_1+s_2^2-6s_2-3)^3(-4s_1s_2+3s_2^2+6s_2-1).\end{aligned}$$

$4s_1+s_2^2-6s_2-3=0$ is the cardioid with cusp at 1, centre at 0; $-4s_1s_2+3s_2^2+6s_2-1=0$ is the bicircular quartic with cusp at 1 which is the locus of cusps of cardioids with centre at 0 and passing through 1. It would seem that there are never more

than 2 cardioids real, for if the second point lies either outside the cardioid

$$4s_1+s_2^2-6s_2-3=0,$$

or inside the quartic

$$-4s_1s_2+3s_2^2+6s_2-1=0,$$

there will be none real; if it lies between these curves, there will be 2 real.

PROBLEM (d): Given the cusp and 2 lines.

Take the cusp at 0; the lines as $z+z'=2$ and $z=z't_1+a_1$.

The centre-locus for cardioids with cusp at 0 and touching the line $z+z'=2$ is the parabolic cubic

$$(c+c'-2)^3+54cc'=0.$$

The centre-locus for cardioids with cusp at 0 and touching the line $z=z't_1+a_1$ is the parabolic cubic

$$(c-c't_1+a_1)^3-27cc'a_1t_1=0.$$

These two curves have 9 common intersections. By a repetition of the argument used in Problem (a), it may be shown that 3 of these are always real. And, as a matter of fact, these loci have but 3 real intersections; i. e., there are 3 cardioids with a given cusp and touching 2 lines. See Fig. IX, p. 43, where the 3 cardioids are shown.

PROBLEM (e): Given the cusp, a line and a point.

Take the cusp at 0; the line as $z+z'=2$, and the point as p.

The locus of centres of cardioids with cusp at the origin and touching the line $z+z'=2$ is the nodal cubic

$$(c+c'-2)^3+54cc'=0.$$

The centre-locus for cardioids with 0 for cusp and on the point p is the parabola

$$4cc'pp'=(cp'+c'p-pp')^2.$$

These confocal curves have 6 common intersections, 2 of which seem to be always imaginary; i. e., there seem to be at most 4 real cardioids having a given cusp, passing through a given point and touching a given line.

PROBLEM (f): Given the cusp and 2 points.

Take the cusp at 0; the points as 1 and p.

The centre-locus for cardioids with cusp at 0 and passing through 1 is the parabola

$$4cc'=(c+c'-1)^2.$$

The centre-locus for cardioids with cusp at 0 and passing through p is the parabola

$$4cc'pp' = (cp' + c'p - pp')^2.$$

These conics have 4 intersections in common; but being confocal parabolas, only 2 of these intersections are real.* There are, then, only 2 real cardioids with a given cusp and 2 given points (Fig. XI).

PROBLEM (g): Given the double tangent and 2 lines.

Take the double tangent as $z+z'=2$; the lines as $z=t_1z'$ and $z=t_2z'+a_2$.

The centre-locus for cardioids with double tangent $z+z'=2$ and touching the line $z=z't_1$ consists of three lines through the intersection of $z+z'=2$ and $z=z't_1$ such that if $t_1=e^{2i\theta}$, the inclinations of the lines to the axis of reals are $\frac{\theta}{3}, \frac{\theta+\pi}{3}, \frac{\theta+2\pi}{3}$, respectively. Similarly, the centre-locus for cardioids with $z+z'=2$ as double tangent and touching $z=t_2z'+a_2$ consists of three lines through the intersection of $z+z'=2$ and $z=z't_2+a_2$ with inclinations $\frac{\varphi}{3}, \frac{\varphi+\pi}{3}, \frac{\varphi+2\pi}{3}$, respectively, where $t_2=e^{2i\varphi}$. These loci intersect in 9 real points; so there are, in general, 9 cardioids with a given double tangent and 2 lines. If, however, the 2 given lines are parallel, there are only 6 proper cardioids; and if 1 line is parallel to the double tangent, there are but 3.

PROBLEM (h): Given the double tangent, a point and a line.

Let the double tangent be $z+z'=2$, 0 the given point, and $z=z't_1+a_1$ the given line.

The centre-locus for cardioids on 0 and touching $z+z'=2$ as a double tangent is the quartic:

$$(2-z-z')^4+4(2-z-z')^3(z+z')+18(2-z-z')^2zz'-27z^2z'^2=0.$$

The centre-locus for cardioids touching $z=z't_1+a_1$ and having $z+z'=2$ as a double tangent consists of the 3 lines as described on page 13. Since a double tangent and a centre determine a cardioid, we should expect the number of cardioids fulfilling the required conditions to be 12, as given by the intersections of the quartic and the 3 lines. However, because of the inclination of the parabolic branches of the quartic and the 60°-inclination of

* Cf. Charlotte Scott, *Modern Analytical Geometry*, N. Y., 1894, p. 77.

the lines towards each other, there are always at least 2, and possibly 4, of the intersections imaginary. Hence, there are at most 10, and possibly only 8, real cardioids with a given point, a given double tangent and touching a given line.

PROBLEM (i): Given the double tangent and 2 points.

As a preliminary statement, let us remark that the 2 points must be both on the same side of the double tangent; otherwise, there would be no real solution.

Let $z+z'=2$ be the given double tangent; 0 and p the given points.

The cusp-locus for cardioids with $z+z'=2$ as double tangent and going through the origin is the quartic

$$4(2-k-k')^2kk'=[-(k+k')(2-k-k')+kk']^2$$

The cusp-locus for cardioids with double tangent $z+z'=2$ and passing through p is the quartic

$$4(2-k-k')^2\,(k-p)(k'-p')=[(2-k-k')(p'-k'+p-k)\\ +(p-k)(p'-k')]^2$$

These two curves have doubly parabolic branches which are respectively parallel. Moreover, they each consist of a cuspidal and a non-cuspidal part, which we may designate for the two curves (Q, Q^1) as K, N, K^1, N^1, respectively.

Since a double tangent and a cusp uniquely determine a cardioid, the number of cardioids satisfying the required conditions is given by the number of intersections of the quartics Q and Q^1. There are 16 such intersections. Of these, however, 4 lead to degenerate curves the cusps of which are on the line at infinity at the points of intersection of the two pairs of parallel parabolic branches. Let us now determine how many of the 12 remaining intersections can be real.

N and N^1, K and K^1 can each intersect but once; N and K^1, K and N^1 may have 0, 1, or 2 intersections. Moreover, p may be so placed as to combine the maximum number of intersections of the cuspidal and non-cuspidal parts of the respective curves (see Fig. V, p. 43). Thus, there are, at most, 6 real proper cardioids going through 2 points and touching a given double tangent.

Here conclude the simpler problems, in each of which the data include one of the fixed elements of the cardioid. We shall now consider a few more difficult problems in which lines and points form the data.

More Difficult Problems.

I. When 3 Conditions Are Given.

PROBLEM (7): Given 3 lines; find the locus of centres.

Let the 3 given lines be $z = z't_i + a_i$ $(i = 1, 2.3)$.

Since each of the lines is to be tangent to the cardioid, we have the identities:

$$z - z't^3 - z_o - 3at + 3a't^2 + z'_o t^3 \equiv z - z't_i - a_i$$

whence, $$t^3 = t_i \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and $$z_o + 3at - a't^2 - z'_o t^3 = a_i \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1), we get $t = \sqrt[3]{t_i} = \tau_i,\ \omega\tau_i,\ \omega^2\tau_i$.

Substituting the value $t = \tau_i$ in (2), we get the three equations:

$$\begin{aligned} z_o + 3a\tau_1 - 3a'\tau_1^2 - z'_o\tau_1^3 &= a_1 \\ z_o + 3a\tau_2 - 3a'\tau_2^2 - z'_o\tau_2^3 &= a_2 \\ z_o + 3a\tau_3 - 3a'\tau_3^2 - z'_o\tau_3^3 &= a_3 \end{aligned}$$

Eliminating a, a' from these 3 equations, we obtain the line

$$\begin{vmatrix} z_o - z'_o\tau_1^3 - a_1 & \tau_1 & \tau_1^2 \\ z_o - z'_o\tau_2^3 - a_2 & \tau_2 & \tau_2^2 \\ z_o - z'_o\tau_3^3 - a_3 & \tau_3 & \tau_3^2 \end{vmatrix} = 0, \text{ as part of the}$$

locus. Since we can choose each of the 3 values of $t (t = \tau_i, \omega\tau_i, \omega^2\tau_i)$ in 3 ways, there are 27 different combinations or 27 different sets of 3 equations in z_o, z'_o, a, a'. But these lead to only 9 distinct lines, arising thus:

The combinations $\left\{\begin{matrix} \tau_1, & \tau_2, & \tau_3 \\ \omega\tau_1, & \omega\tau_2, & \omega\tau_3 \\ \omega^2\tau_1, & \omega^2\tau_2, & \omega^2\tau_3 \end{matrix}\right\}$ give the same line, which we shall designate as L_1.

$\left.\begin{matrix} \omega\tau_1, & \omega^2\tau_2, & \tau_3, \\ \omega^2\tau_1, & \tau_2, & \omega\tau_3 \\ \tau_1, & \omega\tau_2, & \omega^2\tau_3 \end{matrix}\right\}$ give L_2; $\quad$ $\left.\begin{matrix} \tau_1 & \omega^2\tau_2, & \omega\tau_3 \\ \omega\tau_1, & \tau_2, & \omega^2\tau_3 \\ \omega^2\tau_1, & \omega\tau_2, & \tau_3 \end{matrix}\right\}$ give L_3;

$\left.\begin{matrix} \tau_1, & \tau_2, & \omega\tau_3 \\ \omega\tau_1, & \omega\tau_2, & \omega^2\tau_3 \\ \omega^2\tau_1, & \omega^2\tau_2, & \tau_3 \end{matrix}\right\}$ give L_4; $\quad$ $\left.\begin{matrix} \tau_1, & \tau_2, & \omega^2\tau_3 \\ \omega\tau_1, & \omega\tau_2, & \tau_3 \\ \omega^2\tau_1, & \omega^2\tau_2, & \omega\tau_3 \end{matrix}\right\}$ give L_5;

$\left.\begin{matrix} \tau_1, & \omega\tau_2, & \tau_3 \\ \omega\tau_1, & \omega^2\tau_2, & \omega\tau_3 \\ \omega^2\tau_1, & \tau_2, & \omega^2\tau_3 \end{matrix}\right\}$ give L_6; $\quad$ $\left.\begin{matrix} \tau_1, & \omega^2\tau_2, & \tau_3 \\ \omega\tau_1, & \tau_2, & \omega\tau_2 \\ \omega^2\tau_1, & \omega\tau_2, & \omega^2\tau_3 \end{matrix}\right\}$ give L_7;

$\left.\begin{matrix} \omega\tau_1, & \tau_2, & \tau_3 \\ \omega^2\tau_1, & \omega\tau_2, & \omega\tau_3 \\ \tau_1, & \omega^2\tau_2, & \omega^2\tau_3 \end{matrix}\right\}$ give L_8; $\quad$ $\left.\begin{matrix} \omega^2\tau_1, & \tau_2, & \tau_3 \\ \tau_1, & \omega\tau_2, & \omega\tau_3 \\ \omega\tau_1, & \omega^2\tau_2, & \omega^2\tau_3 \end{matrix}\right\}$ give L_9.

The directions of these lines are determined by the ratio of the coefficient of z_o to the coefficient of z'_o in the respective equations. The coefficient of z_o for L_1 is

$$\begin{vmatrix} 1 & \tau_1 & \tau_1^2 \\ 1 & \tau_2 & \tau_2^2 \\ 1 & \tau_3 & \tau_3^2 \end{vmatrix} = (\tau_1-\tau_2)(\tau_2-\tau_3)(\tau_3-\tau_1).$$

The coefficient of z'_o for the same line is

$$\begin{vmatrix} -\tau_1^3 & \tau_1 & \tau_1^2 \\ -\tau_2^3 & \tau_2 & \tau_2^2 \\ -\tau_3^3 & \tau_3 & \tau_3^2 \end{vmatrix} = -\tau_1\tau_2\tau_3(\tau_1-\tau_2)(\tau_2-\tau_3)(\tau_3-\tau_1).$$

The ratio of the coefficients of z_o and z'_o is $1:-\sigma_3$. Likewise, it will be found for L_2 and L_3 that the ratio of the coefficients of z_o : and z'_o is $1:-\sigma_3$; i. e., L_1, L_2, L_3 are parallel.

Similarly, L_4, L_6, L_8 are parallel with directions determined by the ratio $1:-\omega\sigma_3$; and L_5, L_7, L_9 are also parallel with directions determined by $1:-\omega^2\sigma_3$. Now, if θ, φ, ψ be the angles these three sets of lines make with the real axis, then

$$e^{2i(\varphi-\theta)}=\omega$$

$$e^{2i(\varphi-\theta)}=1=e^{\pm 2\pi i}$$

$$\varphi-\theta=\pm\frac{\pi}{3}$$

Similarly, $$\theta-\psi=\pm\frac{\pi}{3}$$

and $$\varphi-\psi=\pm\frac{\pi}{3};$$

i. e., the sets of parallel lines are inclined at angles of 60° towards one another.

Thus, the locus of centres of cardioids touching 3 given lines consists of 3 sets of parallel lines, inclined to each other at angles of 60°. (See Fig. VI.)

Let us endeavor to interpret this locus geometrically. Designating the 3 given lines as l_1, l_2, l_3, respectively, of which l_1 and l_2 intersect at an angle α, let us consider any one cardioid touching these given lines. This cardioid possesses a cusp which may have any one of three positions relative to l_1 and l_2 (its relation to l_3 being, for the present, left out of account) according as the cuspidal rays to the points of tangency make an angle of $\frac{2}{3}\alpha$,

$\frac{2}{3}\alpha+120°$, or $\frac{2}{3}\alpha+240°$. With each of these three cusps is associated a definite centre which, as the variable cardioid assumes positions fulfilling the required conditions, traces out one of the nine lines found as the centre locus. Moreover, each of these three lines forms an angle of 120° (or 60°) with the others. Further, combining each of these three positions relative to l_1, l_2 with the three possible relations with l_3, we obtain the complete locus of 9 lines as above.

PROBLEM (8): Given 2 lines and a point; find the locus of centres.

Let $\left\{\begin{aligned} z&=t_1^2 z' \\ z&=\frac{z'}{t_1^2}\end{aligned}\right\}$ be the given lines, and p the given point.

The identification of these lines with a tangent to the cardioid leads to the equations:

$$t'^3=t_1^2;\ t'=t_1^{\frac{2}{3}},\ \omega t_1^{\frac{2}{3}},\ \omega^2 t_1^{\frac{2}{3}} \qquad (1)$$

$$z_o-z'_o t'^3+3at'-3a't'^2=0 \qquad (2)$$

$$\tau^3=\frac{1}{t_1^2};\ \tau=t_1^{-\frac{2}{3}},\ \omega t_1^{-\frac{2}{3}},\ \omega^2 t_1^{-\frac{2}{3}} \qquad (3)$$

$$z_o-z'_o\tau^3+3a\tau-3a'\tau^2=0 \qquad (4)$$

We have, also, the equations:

$$p-z_o-2at+a't^2=0 \qquad (5)$$
$$(z'_o-p')t^2+2a't-a=0 \qquad (6)$$

The elimination of t, t', τ, a, a' from these six equations will give the required locus.

From (2), (4), we obtain

$$3a=\frac{-z_o s_1-z'_o s_2^2}{s_2} \qquad \text{where } \left\{\begin{aligned} s_1&=t'+\tau \\ s_2&=t'\tau\end{aligned}\right.$$
$$3a'=\frac{-z_o-z'_o s_1 s_2}{s_2}$$

From (5), (6):

$$\begin{vmatrix} a' & -2a & p-z_o & 0 \\ 0 & a' & -2a & p-z_o \\ z'_o-p' & 2a' & -a & 0 \\ 0 & z'_o-p' & 2a' & -a \end{vmatrix} = 0.$$

Substitution of the values of a, a' in this determinant furnishes a quartic as part of the sought locus. Since there are 3 values each for t' and τ, there seem to be 9 quartics in z_o, z'_o, s_1, s_2. But these 9 reduce to 3; for the combinations of t', τ, divide off in sets of 3, producing only 3 distinct developments of the determinant. These combinations are, in fact, the following:

$$\left.\begin{matrix} t', & \tau \\ \omega t', & \omega\tau \\ \omega^2 t', & \omega^2\tau \end{matrix}\right\}, \qquad \left.\begin{matrix} \omega t', & \omega^2\tau \\ \omega^2 t', & \tau \\ t', & \omega\tau \end{matrix}\right\}, \qquad \left.\begin{matrix} \omega^2 t', & \omega\tau \\ t', & \omega^2\tau \\ \omega t', & \tau \end{matrix}\right\}.$$

Now the only terms in the development involving a or a' are those in a^3, a'^3, aa', $a^2a'^2$. The effect of replacing t', τ by $\omega t'$, $\omega\tau$, respectively, in the values of a and a' is to multiply the equation by $\omega^3(=1)$. Similarly, the substitution of $\omega^2 t'$, $\omega^2\tau$ for t' and τ, respectively, multiplies the equation by 1. Thus, the combinations in the first set lead to the same line. Likewise, those in the other 2 sets lead to but 2 other distinct lines.

These 3 sets of combinations lead to the following values of s_1, s_2:

$$\begin{cases} s_1 = 2\cos\frac{2}{3}\theta \\ s_2 = 1 \end{cases}$$

$$\begin{cases} s_1 = 2\cos\left(\frac{2}{3}\theta + \frac{2}{3}\pi\right) \\ s_2 = 1 \end{cases}$$

$$\begin{cases} s_1 = 2\cos\left(\frac{2}{3}\theta + \frac{4}{3}\pi\right) \\ s_2 = 1 \end{cases}$$

where $t_1 = e^{i\theta}$. Hence, s_2 may always be taken equal to 1, and the development is

$$(z_o + z'_o s_1)^2 (z_o s_1 + z'_o)^2 - 4[(z_o - p)(z_o + z' s_1)^3 + (z'_o - p')(z_o s_1 + z'_o)^3]$$
$$+ 18(z_o - p)(z'_o - p')(z_o + z'_0 s_1)(z_o s_1 + z'_o) - 27(z_o - p)^2 (z'_o - p')^2 = 0,$$

where s_1 has one of the values above. This equation can be put in the form

$$(s_1^2 - 4)(z_o^2 - s_1 z_o z'_o + z'^2_o)^2 + f(z_o, z'_o, p, p')^3 = 0,$$

which indicates 2 parabolic branches. These branches are inclined at angles $\pm\frac{\theta}{3}$, $\pm\left(\frac{\theta}{3} + \frac{\pi}{6}\right)$, $\pm\left(\frac{\theta}{3} + \frac{\pi}{3}\right)$ for the three quartics, respectively. Moreover, the locus has a cusp at the point where the variable cardioid comes into the position where the fixed point

is its cusp. Thus, the centre locus consists of 3 quartics with properties as described above. The curves are of the form shown in Fig. VI.

Geometrically interpreted, it appears that the 3 quartics are traced out by the centres of those cardioids (touching the given lines and on the given point) of which the cuspidal rays to the points of tangency are inclined towards each other at angles $\frac{2}{3}\theta$, $\frac{2}{3}\theta+120°$, $\frac{2}{3}\theta+240°$, respectively, where θ is the angle between the given lines.

The specialization of this problem, arising when p is the point of tangency on one of the given lines, proves instructive. Let us first view the problem from a geometric standpoint and see what *a priori* information we can derive therefrom. Consider a case where 3 lines, l_1, l_2, l_3, are given, of which l_1 and l_2 are fixed in position, while l_3 rotates about a fixed point p on l_1. For any position of the variable line, the centre-locus consists of 3 sets of 3 lines as found in Problem (7). The limiting position of l_3 is that in which it comes into coincidence with l_1. Then p becomes the point of tangency on l_1, and the conditions become equivalent to the data of this specialized problem. Thus, we may expect sets of lines in the centre-locus. Will the 3 sets appear? To answer this, recall the fact that the 3 sets of lines in the centre-locus of Problem (7) are associated with cardioids such that the cuspidal rays to the points of tangency make angles equal to $\frac{2}{3}\theta$, $\frac{2}{3}\theta+120°$, $\frac{2}{3}\theta+240°$, respectively, where θ is the angle between the tangents. In the case under consideration θ, the angle between l_1 and l_3, is 0; and the cuspidal rays to the point of tangency make an angle 0° only. Thus, the 3 sets of 3 lines seem to reduce to one set of 3. What happens to the other 2 sets of lines? They are evidently associated with cardioids such that the cuspidal rays to the points of tangency make angles of 120° and 240°, respectively; i. e., with cardioids of which the tangents make angles of 180° or 360°, that is, with cardioids with parallel tangents. Thus, the other 2 sets of lines belong, not to the case where l_3 and l_1 are coincident, but to the case where they are parallel.*

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* Cf. the theorem indicated on p. 8.

Let us now consider the problem analytically.

Let us take $\left\{\begin{array}{l} z=t_1^2z' \\ z=\dfrac{z'}{t_1^2} \end{array}\right\}$ as the given lines, and p as the point on $z=t_1^2z'$. Let us find the locus of centres.

The identification of the given lines with a tangent leads to equations (1), (2), (3), (4) of Problem (8). Since p is the point of tangency on $z=t_1^2z'$ and is also on the variable cardioid, we have

$$p=z_0+2at'-a't'^2 \qquad (5)$$

The elimination of a, a' from (2), (4) and (5) gives the line

$$\begin{vmatrix} z_0-z'_0\tau^3 & 3\tau & 3\tau^2 \\ z_0-z'_0t'^3 & 3t' & 3t'^2 \\ z_0-p & 2t' & t'^2 \end{vmatrix}=0$$

as the required locus.

By development of the determinant, this line is found to be

$$z_0(2t'^2\tau-t'\tau^2-t'^3)-z'_0t'^2\tau(2t'^2\tau-t'\tau^2-t'^3)-3pt'\tau(t'-\tau)=0.$$

Although there are 9 possible combinations of t' and τ, there are not the same number of lines in the locus. In fact, since $t'=\dfrac{1}{\tau}$, $\dfrac{\omega}{\tau}$, or $\dfrac{\omega^2}{\tau}$, and the combinations of t' and τ are all of the third degree, there are only 3 distinct lines with clinants $t_1^{\frac{2}{3}}$, $\omega t_1^{\frac{2}{3}}$, $\omega^2t_1^{\frac{2}{3}}$, respectively; i. e., the 3 lines are inclined towards each other at angles of 60° (or 120°).

PROBLEM (9): Given 2 points and 1 line; find the locus of centres.

Take the line as $z+z'=2$, the points as 0 and p. Let the equation of the cardioid be

$$z=z_0+2at-a't^2.$$

Since the cardioid touches $z+z'=2$, we have the identity:

$$z-z_0-(z'-z'_0)t^3-3at-3a't^2\equiv z+z'-2;$$

whence,

$$t^3=-1;\ t=-1,\ -\omega,\ -\omega^2 \qquad (1)$$

$$z_0-z'_0t^3+3at-3a't^2=2 \qquad (2)$$

Since p and 0 are points on the cardioid, we have

$$p=z_0+2at'-a't'^2 \qquad (3)$$

$$p'=z'_0+\frac{2a'}{t'}-\frac{a}{t'^2} \qquad (4)$$

$$0 = z_0 + 2a\tau - a'\tau^2 \dots\dots\dots\dots\dots\dots (5)$$

$$0 = z'_o + \frac{2a'}{\tau} - \frac{a}{\tau^2} \dots\dots\dots\dots\dots\dots (6)$$

The elimination of t, t', τ, a, a' from these 6 equations will give the required centre-locus.

Eliminating t' from (3), (4), we obtain the equation:

$$\begin{vmatrix} -a' & 2a & z_o - p & 0 \\ 0 & -a' & 2a & z_o - p \\ p' - z'_o & -2a' & a & 0 \\ 0 & p' - z'_o & -2a' & a \end{vmatrix} = 0.$$

Developed, this is

$$3a^2a'^2 - 4a^3(z'_o - p') - 4a'^3(z_o - p) - 6aa'(z_o - p)(z'_o - p') \\ + (z_o - p)^2(z'_o - p')^2 = 0 \dots\dots (7)$$

The elimination of τ from (5), (6) leads to

$$3a^2a'^2 - 4a^3z_o - 4a'^3z'_o - 6aa'z_oz'_o + z_o^2z'^2_o = 0 \dots\dots (8)$$

From (1), employing the value $t = -1$,

$$z_o + z'_o - 3a - 3a' = 2 \dots\dots\dots\dots\dots\dots (9)$$

From (7), (8), (9), a, a' have still to be eliminated. Let us transform to Cartesian co-ordinates by replacing a by $a+ib$, z by $x+iy$, p by $p+iq$, with a corresponding substitution for their conjugates. Equation (8) becomes

$$3(a^2+b^2)^2 - 8(a^3x - 3b^2ax - 3a^2by + b^3y) - 6(a^2+b^2)(x^2+y^2) \\ + (x^2+y^2)^2 = 0$$

Equation (7) becomes a similar equation in which $x-p$, $y-q$ replace x and y, respectively. Equation (9) gives $a = \frac{x-1}{3}$. Substituting the value of a in (7) and (8), we obtain the quartics in b:

$$27b^4 - 72b^3y + 18b^2(4x^2 - 9y^2 - 2x - 11) + 72by(x-1)^2 + f(x, y)^4 = 0$$

$$27b^4 - 72b^3(y-q) + 18b^2f(x, y, p, q)^2 + 72b(y-q)(x-1)^2 \\ + \Phi(x, y, p, q)^4 = 0$$

The elimination of b leads to an 8-row determinant, which, developed, is an equation of the 12th degree in x and y. Thus, the centre-locus is a 12-ic. We shall not attempt to discuss the singularities of this locus.

Let us now find the centre-locus for the case where one of the points is on the line; that is, let the data be the line $z+z'=2$ and the points 0 and p, of which p is on the given line.

From p. 29, we have $t=-1,\ -\omega,\ -\omega^2$. . . . . (1)

$$z_o-z'_o\,t^3+3at-3a't^2=2 \qquad (2)$$

$$3a^2a'^2-4a^3z_o-4a'^3z'_o-6aa'z_oz'_o+z_o^2z'^2_o=0 \qquad (3)$$

Since p is the point of tangency on the cardioid,

$$p=z_o+2at-a't^2 \qquad (4)$$

From (2) and (4),

$$3at=3p-2z_o+z'_ot^3-2$$

$$\frac{3a'}{t}=3p'-2z'_o+\frac{z_o}{t^3}-2$$

Since a, a', occur in (3) only in the combinations $a^2a'^2$, a^3, a'^3, aa', any of the values $t=-1$, $-\omega$, or $-\omega^2$ will lead to the same result. Employing any one of these values, we obtain a quartic in z_o, z'_o as the sought centre-locus.

We might have expected such a result, for the data from the limiting case for 2 lines and a point,* where the lines coincide and their point of intersection becomes the point of tangency. Moreover, we can account for the reduction in the number of quartics in the locus by recognizing that the 2 quartics which do not appear here belong, not to the case where the 2 lines coincide, but to the case where they are parallel.

PROBLEM (10): Given 3 points; find the locus of cusps.

Let the given points be 0, p, q, and the map-equation of the cardioid be

$$z=k-a(1-t)^2.$$

If t_1 be the value of t at 0,

$$\left.\begin{aligned} k-a(1-t_1)^2&=0\\ k'-a'\left(\frac{1-t_1}{t_1}\right)^2&=0\end{aligned}\right\};$$

whence, $$(kk'-a'k-ak')^2=4aa'kk' \qquad (1)$$

Similarly, since the cardioid is on p,

$$[(k-p)(k'-p')-a'(k-p)-a(k'-p')]^2=4aa'(k-p)(k'-p') \qquad (2)$$

Since q is a third point on the cardioid, we have

$$[(k-q)(k'-q')-a'(k-q)-a(k'-q')]^2=4aa'(k-q)(k'-q') \qquad (3)$$

We have to eliminate a, a', from (1), (2), (3).

---

* Cf. Problem (8), p. 26.

Transforming to Cartesian co-ordinates and letting

$$\left.\begin{aligned} k&=X+iY\\ a&=A+iB\\ p&=P+iQ\\ q&=U+iV \end{aligned}\right\},$$

the equations become

$$(X^2+Y^2-2AX-2BY)^2=4\,(A^2+B^2)(X^2+Y^2)\ldots\ldots(1)$$

$$[(X-P)^2+(Y-Q)^2-2A(X-P)-2B\,(Y-Q)]^2$$
$$=4(A^2+B^2)\,[(X-P)^2+(Y-Q)^2]\ldots(2)$$

$$[(X-U)^2+(Y-V)^2-2A(X-U)-2B\,(Y-V)]^2$$
$$=4(A^2+B^2)\,[(X-U)^2+(Y-V)^2]\ldots(3)$$

Interpreting A and B as Cartesian co-ordinates, these equations represent parabolas with 0 as focus. We seek the condition that the three curves have a common point.

Since (1) and (2) [also (1) and (3)] are confocal parabolas, they have 3 known common lines, of which two are imaginary (the lines from I and J); and the third (the line at infinity), real. Their fourth common line is, therefore, rationally obtainable and, therefore, their self-conjugate triangle. This triangle will have 1 real vertex and 2 imaginary ones; and the one real pair of common chords of the two parabolas will pass through the real vertex. If we find the equations of the line-pairs common to parabolas (1) and (2), (1) and (3), respectively, and require that, with properly chosen sign, they be simultaneously true, their solution will yield the co-ordinates of the point which is to be common to the 3 parabolas. The substitution of these values in the equation of one of the parabolas will give the cusp-locus sought.

Parabola (1) in line co-ordinates (using ξ, η, ζ corresponding to a, a', w) is

$$\eta\zeta k'+\zeta\xi k+\xi\eta kk'=0.$$

Parabola (2) is

$$\eta\zeta(k'-p')+\zeta\xi(k-p)+\xi\eta(k-p)(k'-p')=0.$$

Their fourth common line is

$$\frac{a}{kp'(k-p)}-\frac{a'}{k'p(k'-p')}-\frac{1}{kp'-k'p}=0.$$

The one real diagonal point of the 4-line is $[kp'(k-p),\ -k'p(k'-p'),\ kp'-k'p]$. We can now find the equation of the sought line-pair by writing the pencil $U+\lambda V=0$ built up on the 2 parabolas, and

determine λ so that the resulting curve shall contain this point.

We easily find as the equation of the line-pair

$$(k-p)(k'-p')(kk'-ak'-a'k)^2=kk'[(k-p)(k'-p')-a(k'-p')-a'(k-p)]^2$$

Using Cartesians, the line-pair is

$$[(X-P)^2+(Y-Q)^2\ [(X^2+Y^2-2AX-2BY)^2 = (X^2+Y^2)[\overline{X-P}^2+\overline{Y-Q}^2-2A\overline{X-P}-2B\overline{Y-Q}]^2,$$

Now, if we let $\left.\begin{array}{l} X^2+Y^2=\rho^2 \\ \overline{X-P}^2+\overline{Y-Q}^2=\sigma^2 \end{array}\right\}$, where ρ, σ are taken to be the positive square roots, the 2 lines are

$$\sigma(\rho^2-2AX-2BY)=\pm\rho[\sigma^2-2A(X-P)-2B(Y-Q)]\ldots.(4)$$

Similarly, the line-pair common to the parabolas (1) and (3) is

$$\tau(\rho^2-2AX-2BY)=\mp\rho[\tau^2-2A(X-U)-2B(Y-V)]\ldots.(5)$$

where $\tau=+\sqrt{(X-U)^2+(Y-V)^2}$

If the three curves are to have a common point, equations (4) and (5), with properly chosen sign, must be simultaneously true.

In terms of A and B, these equations are

$$2A[P\rho-X(\sigma+\rho)]+2B[Q\rho-Y(\sigma+\rho)]+\sigma\rho(\sigma+\rho)=0.\ \ldots(4)$$
$$2A[U\rho-X(\tau+\rho)]+2B[V\rho-Y(\tau+\rho)]+\tau\rho(\tau+\rho)=0.\ \ldots(5)$$

The solution of these two equations and the substitution of the resulting values of A and B in the equation of one of the parabolas will yield the cusp-locus sought.

From (4), (5)

$$A=-\frac{\rho\begin{vmatrix}\sigma(\sigma+\rho) & [Q\rho-Y(\sigma+\rho)]\\ \tau(\tau+\rho) & [V\rho-Y(\tau+\rho)]\end{vmatrix}}{2\begin{vmatrix}[P\rho-X(\sigma+\rho)] & [Q\rho-Y(\sigma+\rho)[\\ [U\rho-X(\tau+\rho)] & [V\rho-Y(\tau+\rho)]\end{vmatrix}}$$

$$B=\frac{\rho\begin{vmatrix}\sigma(\sigma+\rho) & [P\rho-X(\sigma+\rho)]\\ \tau(\tau+\rho) & [U\rho-X(\tau+\rho)]\end{vmatrix}}{2\begin{vmatrix}[P\rho-X(\sigma+\rho)] & [Q\rho-Y(\sigma+\rho)]\\ [U\rho-X(\tau+\rho)] & [V\rho-Y(\tau+\rho)]\end{vmatrix}}$$

Designating the determinants in the numerators as D_1 and D_2, respectively, and the denominator determinant as D_o,

$$A^2+B^2=\frac{\rho^2}{4}\frac{(D_1^2+D_2^2)}{D_o^2}$$

$$AX+BY=\frac{\rho}{2}\frac{(D_1X+D_2Y)}{D_o}$$

$$\begin{aligned} D_1^2+D_2^2=\rho^2[\sigma^2(\rho+\sigma)^2(U^2+V^2)+(\sigma-\tau)^2(\rho+\sigma)^2(\rho+\tau)^2 \\ +\tau^2(\rho+\tau)^2(P^2+Q^2)-2\sigma\tau(\rho+\sigma)(\rho+\tau)(PU+QV)] \\ -2\rho(\sigma+\rho)(\rho+\tau)(\sigma-\tau)[\sigma(\rho+\sigma)(UX+VY)-\tau(\rho+\tau)(PX+QY)] \end{aligned}$$

$$D_1X+D_2Y=\rho\left\{-\sigma(\rho+\sigma)\begin{vmatrix} X & Y \\ U & V \end{vmatrix}+\tau(\rho+\tau)\begin{vmatrix} X & Y \\ P & Q \end{vmatrix}\right\}$$

D_o may be expressed thus:

$$\rho\left\{\rho\begin{vmatrix} P & Q \\ U & V \end{vmatrix}-(\rho+\sigma)\begin{vmatrix} X & Y \\ U & V \end{vmatrix}+(\rho+\tau)\begin{vmatrix} X & Y \\ P & Q \end{vmatrix}\right\}$$

The substitution of these values in

$$[X^2+Y^2-2(AX+BY)]^2=4(A^2+B^2)(X^2+Y^2)$$

gives a result which, after division by ρ^2, is of the form

$$E+F\rho\sigma+G\rho\tau+H\sigma\tau=0,$$

where E is of the sixth degree in X, Y; and F, G, and H are of the fourth degree in the same variables. Rationalizing:

$$(E+H\sigma\tau)^2=(F\rho\sigma+G\rho\tau)^2$$
$$E^2+H^2\sigma^2\tau^2-F^2\rho^2\sigma^2-G^2\rho^2\tau^2=(2FG\rho^2-2EH)\sigma\tau$$

Squaring again,

$$\begin{aligned} E^4+F^4\rho^4\sigma^4+G^4\rho^4\tau^4+H^4\sigma^4\tau^4-2[E^2F^2\sigma^2\rho^2+E^2G^2\rho^2\tau^2+H^2G^2\rho^2\sigma^2\tau^4 \\ +E^2H^2\sigma^2\tau^2+F^2G^2\rho^4\sigma^2\tau^2+F^2H^2\sigma^4\tau^2\rho^2]+8EFGH\rho^2\sigma^2\tau^2=0 \end{aligned}$$

Thus, it appears that the required locus is of the 24th degree. We shall not attempt to discuss the singularities of the curve beyond remarking that, since there are 4 cardioids with a given cusp and on 2 given points, it would seem that this locus has quadruple points at each of the 3 given points.

Further, we may reduce the unrationalized equation to a convenient form thus:

$$X^2+Y^2-2(AX+BY)=$$

$$\frac{\rho\left\{\rho^2\begin{vmatrix} P & Q \\ U & V \end{vmatrix}-(\rho^2-\sigma^2)\begin{vmatrix} X & Y \\ U & V \end{vmatrix}+(\rho^2-\tau^2)\begin{vmatrix} X & Y \\ P & Q \end{vmatrix}\right\}}{\rho\begin{vmatrix} P & Q \\ U & V \end{vmatrix}-(\rho+\sigma)\begin{vmatrix} X & Y \\ U & V \end{vmatrix}+(\rho+\tau)\begin{vmatrix} X & Y \\ P & Q \end{vmatrix}}$$

This numerator can be reduced to

$$-\rho \begin{vmatrix} \rho^2 & x & y \\ p^2+q^2 & p & q \\ u^2+r^2 & u & v \end{vmatrix}$$, which, put in the form

$$-\rho \begin{vmatrix} x^2+y^2 & x & y & 1 \\ p^2+q^2 & p & q & 1 \\ u^2+v^2 & u & v & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$, and equated to 0, is the product of

the equations of the circle through the three given points, 0, p, q, and the lines $0I$, $0J$. Let the numerator be called $-\rho C$. Notin σ that ρ, σ, τ are factors of A^2+B^2, this expression may be designated as $\rho\sigma\tau K$. Thus, the required locus may be put in the form

$$-C=4\rho\sigma\tau K D_o,$$

which shows that the locus intersects each of the lines joining the given points with I and J only at these circular points.

II. When 4 Conditions Are Given.

We may remark that in the problems which follow, the intersections of loci do not furnish the exact number of solutions as is the case in the simpler problems. In these latter, the elements common to curves on both loci are such as to uniquely determine a cardioid; while in the cases about to be considered, the common elements do not so determine a single cardioid.

PROBLEM (j): Given 4 tangents.

Let the 4 given lines be $z=z't_i+a_i$ $(i=1, 2, 3, 4)$. The centre-locus for cardioids touching $z=z't_i+a_i$ $(i=1, 2, 3)$ consists of the 9 lines as described on page 25. The centre-locus for cardioids touching $z=z't_i+a_i$ $(i=1, 2, 4)$ consists, similarly, of 9 lines which divide off into 3 sets of parallel lines with directions 60° apart.

It might at first sight seem that there are 81 cardioids touching the four given lines. But, recalling the fact that the locus is in each case generated by the centres of 3 variable cardioids so related to the lines l_1, l_2 that the angles between the cuspidal rays to the points of tangency are $\frac{2}{3}\theta$, $\frac{2}{3}\theta+120°$, $\frac{2}{3}\theta+240°$, respectively, it is readily seen that only the intersections of corresponding lines of the two loci give a common centre for the 4 lines. The lines

correspond in sets of three. Thus, there are 27* cardioids touching 4 given lines.

As a confirmation of this result, we may note that, although the centre-loci for cardioids touching l_1, l_2, l_3 and l_1, l_2, l_4 intersect in 81 points, these intersections will not give 81 cardioids unless there is but a single cardioid with a given centre and touching 2 given lines. There are actually 3 such cardioids, which suggests the reduction of the number of cardioids touching 4 given lines to 27.

PROBLEM (k): Given 3 lines and a point.

Let the given lines be $z=z't_i+a_i$ $(i=1, 2, 3)$, where $t_2=\frac{1}{t_1}$ and $a_1=a_2=0$; take p as the given point.

The centre-locus for cardioids touching $z=z't_i+a_i$ $(i=1, 2, 3)$ consists of three sets of parallel lines similar to those described in Problem (7). The centre-locus for cardioids touching $z=z't_i+a_i$ $(i=1, 2)$ and passing through p consists of the 3 quartics discussed on page 27.

Although these loci intersect in 108 points, there are, by no means, that many cardioids. For, with each centre on these loci is associated a definite cusp; and it is only the intersections of centre-loci corresponding to the same cusp that give a common centre for a cardioid on the given point and touching the given lines. Now these loci pair off in such a way that to each quartic correspond 3 of the 9 lines;† so that there are 36 centres of cardioids, and, therefore, 36 cardioids satisfying the given conditions. Fig. VI illustrates the case where the given lines form an equilateral triangle.

When the point is on one of the lines, there are but 9 cardioids. For, designating the lines as l_1, l_2, l_3, with p on l_1, the centre-locus for cardioids on l_1, p_1, l_2 consists of 3 lines;‡ likewise, the centre-locus for cardioids on l_1, p_1, l_3 consists of 3 lines. These loci have 9 common intersections, which yield centres for cardioids on l_1, l_2, l_3 and p.

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* Professor Morley has indicated the number as 2^3. Cf. *Trans. Amer. Math. Soc.*, 1900, Vol. 1, p. 114. As above shown, it would seem that there are 3^3.

† Cf. p. 24.

‡ Cf. Problem (8), p. 26.

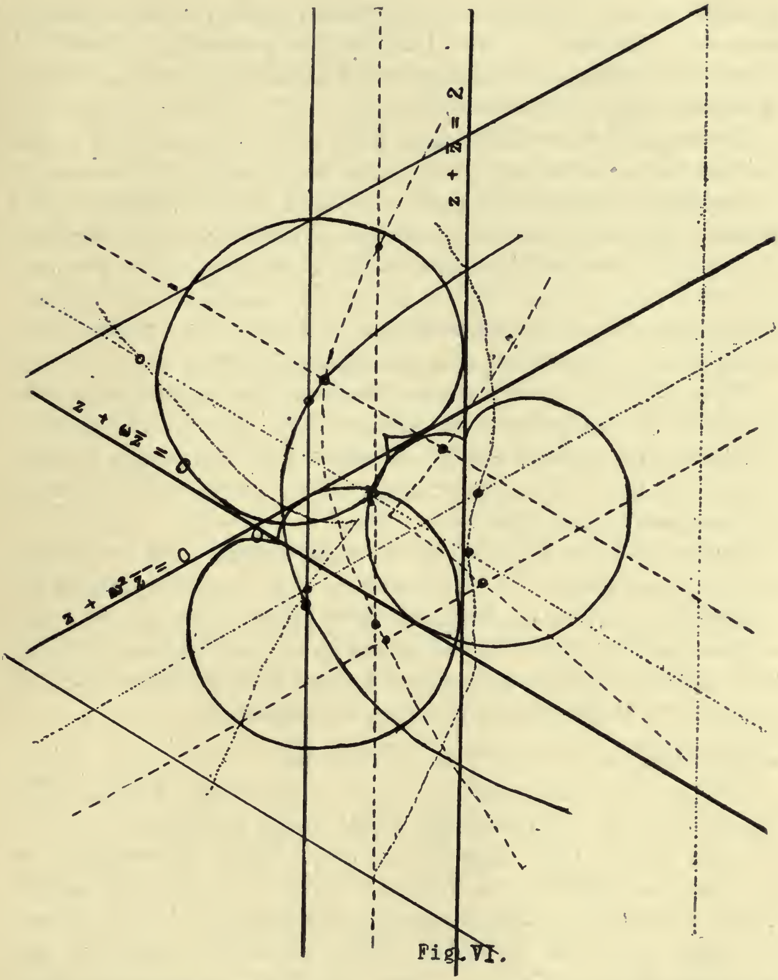

Fig. VI.

Fig. VI.—The loci marked ——, - - - -, ..., are generated by the centers of cardioids of which the cuspidal rays to the points of tangency make angles of 40°, 160°, and 280°, respectively. 3 of the 36 possible cardioids are here shown. For this particular case, there are actually 12 of the cardioids real, as shown by the marked centers.

PROBLEM (1): Given 2 lines and 2 points.

Let the given lines be $z=\tau^2 z'$ and $z=\frac{z'}{\tau^2}$; the given points, p and q.

The centre-locus for cardioids on p and touching $z=\tau^2 z'$, $z=\frac{z'}{\tau^2}$

consists of the 3 quartics discussed on page 27. The centre-locus for cardioids on q and touching the given lines consists of 3 quartics similar to the above, in the equations of which q and q' replace p and p', respectively.

These quartics, which we may designate as Q_1, Q_2, Q_3, Q'_1, Q'_2, Q'_3 for the two loci respectively, pair off in such a way that every intersection of Q_i and Q'_i gives a cardioid on the given lines and points. There are 48 intersections. But Q_i and Q'_i are both doubly parabolic with the same points at infinity. This accounts for 4×3 intersections, which are to be regarded as improper. This leaves 36 proper intersections; i. e., there are 36 cardioids which touch 2 given lines and pass through 2 given points.

If one of the points is on one of the lines, this number reduces to 12. For the centre-locus for cardioids on l_1, l_2, p_1 (on l_1) consists of 3 lines; the centre-locus for cardioids on l_1, p_1, p_2 is a quartic. These loci have 12 common intersections, which yield 12 centres for cardioids on l_1, l_2, p_1, p_2.

Further, if p_2 is on l_2, the number reduces to 6. For the centre-locus for cardioids on l_1, p_1, l_2, consists of 3 lines, as does that for cardioids on l_1, l_2, p_2. Although these loci have 9 common intersections, only 6 are in the finite plane; for since the clinants of the lines in both loci are determined by the angle of intersection of l_1 and l_2,* 3 of the intersections are at infinity.

PROBLEM (m): Given 1 line and 3 points.

Let the given line be $z+z'=2$; the given points, 0, p, q. The centre-locus for cardioids on 0 and p and touching $z+z'=2$ is a curve of the 12th degree; the centre-locus for cardioids on 0, q, and the given line is, likewise, a curve of the 12th degree. These curves have 144 common intersections; so there cannot be more than 144 cardioids which touch the given line and pass through 0, p, and q. There will likely be many less than 144; and the fact that there are 4 cardioids with a given centre, point and line suggests that this number may reduce to 36.

If one of the points is on the given line, there is a further reduction of the number to 12. For, the centres of cardioids on l_1, 0, and p (where p is on l_1) describe a quartic; the centres of cardioids on l_1, p, q, describe a quartic. These loci have 16 common intersections, only 12 of which yield centres of non-degenerate curves.

* Cf. p. 29.

PROBLEM (n): Given 4 points.

Let 0, p, q, r be the given points.

The cusp-locus for cardioids on the points 0, p, q is a curve of the 24th degree; similarly, the cusp-locus for cardioids on 0, p, r is a curve of the same degree. Although these two loci intersect in 24^2 points, there will, by no means, be 24^2 cardioids on the 4 given points. While we shall not attempt to ascertain the exact number, we remark that the fact that there are 4 cardioids with a given cusp and on 2 given points will greatly reduce this 24^2.

We shall conclude with a brief discussion of a problem related to those we have been treating, the solution of which would probably prove of value in the determination of the number of cardioids on 4 points. It has to do with the number of real equilateral triangles which can be inscribed in a given cardioid when 1 vertex is fixed.

Take the given cardioid with its cusp at 0, and at the right of the figure. Its map-equation is $z=-(1-t)^2$. Let this be the standard size.

First, let us consider a triangle of any shape, and let us determine the relations between the vectors to the vertices. Let the vectors to the vertices of a triangle, of which the sides and angles are ρ_1, ρ_2, ρ_3, φ, ψ, θ, respectively, be a, b, c, the terminal end of vector a being at the apex of the angle φ, of vector b at ψ, and of vector c at θ. Then

$$\frac{\rho_2}{\rho_1}(b-c)e^{-i\theta}=-\frac{\rho_2}{\rho_3}(b-a)e^{i\varphi}.$$

$$b(\rho_3 e^{-i\theta}+\rho_1 e^{i\varphi})-c\rho_3 e^{-i\theta}-a\rho_1 e^{i\varphi}=0$$

$$c=-a\frac{\rho_1}{\rho_3}e^{i(\theta+\varphi)}+b[1+\frac{\rho_1}{\rho_3}e^{i(\rho+\varphi)}]$$

Now, let a and b be the points z_1 and z_2 on the given cardioid. Then

$$c=\frac{\rho_1}{\rho_3}e^{i(\theta+\varphi)}(1-t_1)^2-[1+\frac{\rho_1}{\rho_3}e^{i(\theta+\varphi)}](1-t_2)^2$$

If we regard z_1 as fixed, z_2 as variable, the third vertex of the triangle, c, will describe a cardioid with its cusp at the point $\frac{\rho_1}{\rho_3}e^{i(\theta+\varphi)}(1-t_1)^2$, of size $\left|1+\frac{\rho_1}{\rho_3}e^{i(\theta+\varphi)}\right|$, and with orientation-angle $[1+\frac{\rho_1}{\rho_3}e^{i(\theta+\varphi)}]$.

For the case of the equilateral triangle, this becomes particularly simple. Then the third vertex, c, travels on a cardioid with map-equation: $c=\omega(1-t_1)^2+\omega^2(1-t)^2$. This cardioid is of standard size, with an orientation of 60° and its cusp at $\omega(1-t_1)^2$; i. e., the cusp is at the same distance from 0 as is z_1, with its vector turned through an angle of $-60°$. Moreover, this c-cardioid goes through the point z_1 on the original cardioid. Either of two arguments may serve to show this:

1. When the second vertex is made to approach $-(1-t_1)^2$, the third vertex does likewise. Thus, the fixed vertex, z_1, towards which, in the case of an infinitely small triangle, the other two vertices tend, must be on both cardioids.

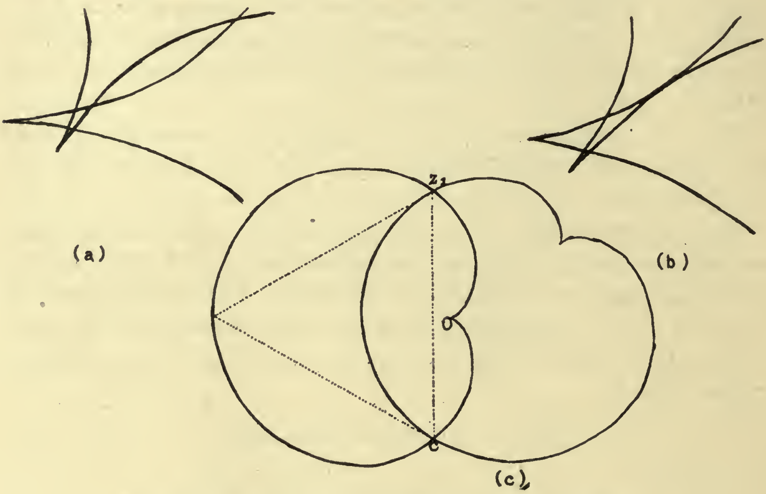

FIG. VII.—The sixth intersection is not shown.

2. Analytically, if in the equation above $t=t_1$, then $c=(\omega+\omega^2)(1-t_1)^2=-(1-t_1)^2$; i. e., z_1 is a point on both cardioids.

If z_1 is given, the number of equilateral triangles is the number of intersections of two cardioids less one, since the infinitely small triangle, all the vertices of which are at z_1, should hardly be regarded as a proper triangle. This number is 7, since the common cusps at I and J account for 8 of the 16 intersections. It happens, however, that 2 of these 7 intersections are always imaginary; so that the maximum number of real proper equilateral triangles inscribable in a given cardioid is 5. However, there are not always 5; there may be but 3, or only 1.

That there may be as many as 5 is easily shown: for, when t is a turn through an angle numerically less than $\pm 30°$, for the point $\omega(1-t_1)^2$ the cusp of the cardioid-locus of the vertex, c, is inside the given cardioid. When t_1, then, is a turn through a very small angle, there will be positions where the number of real intersections will be 5. (See Fig. VII, a). For one value of t_1, the two curves will be tangent, and 2 of the 5 will coincide (Fig. VII, b); while for most values for which t_1 is a turn through an angle greater than 30° (certainly for values of θ between 30° and 330°), there is but 1 real equilateral triangle (Fig. VII, c).

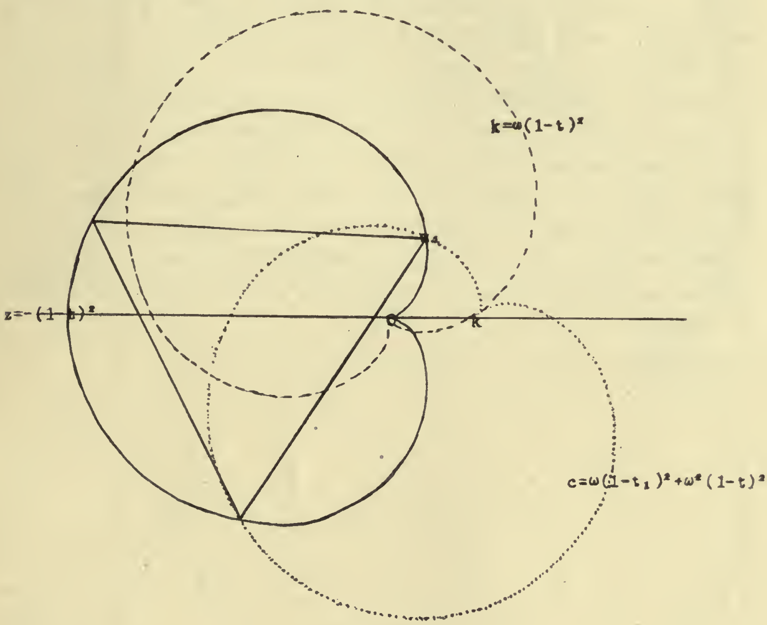

FIG. VIII.

Furthermore, it may be remarked that if we now let z_1 be regarded as variable, the cusp k of the cardioid-locus of the third vertex c is given by $k=\omega(1-t_1)^2=-\omega[-(1-t_1)^2]$; i. e., a cardioid with the same cusp and size as the original, but with an orientation of $-60°$. The relations of these curves is shown in Fig. VIII.

Although we have specialized for the case of equilateral triangles, it is to be observed that the number of triangles of any shape inscribable in a given cardioid is 5, 3, or 1, according to the position of z_1; and that, regarding z_1 as variable, the cusp of the c-cardioid will also describe a cardioid.

We shall now indicate how these considerations may be of

value in determining the number of real cardioids on 4 points. Let a, b, c, d be the vectors to the given points. First, consider the triangle formed by a, b, c. If a be fixed on the standard cardioid and b be allowed to move along the cardioid, c will trace out a second cardioid with relations to the original as indicated on page 39. Similarly, fixing the point a of the triangle formed by a, b, d, on the standard cardioid and allowing b to move along the cardioid, d, also, will trace out a third cardioid with relations to the original as those on page 39, where d replaces c, ρ_4, ρ_5 replace ρ_1, ρ_2 and α, β, δ replace φ, ψ, θ, respectively.

Now a mechanism could be devised by which, after having placed the c-cardioid and the d-cardioid in relation to the original, z_1 is allowed to move along the original curve. The cusps of the two third-vertex-cardioids will also move on cardioids, as shown on page 39. As z_1 moves, the second point, z_2 (b), will not, in general, be the same point for the c- and the d-curve; but such a coincidence will certainly occur. When this does happen, we have a cardioid on the 4 given points. An arrangement could be made by which such a coincidence would be indicated; and a simple registering of these indications will give the number of real cardioids on 4 points.

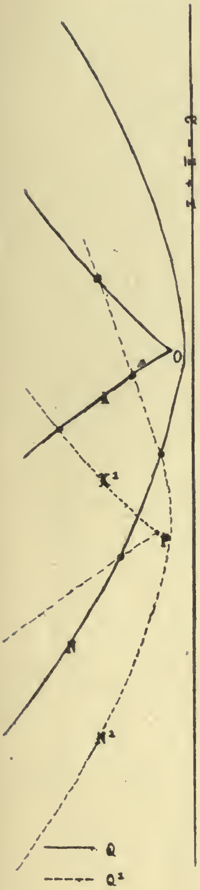

Fig. V.

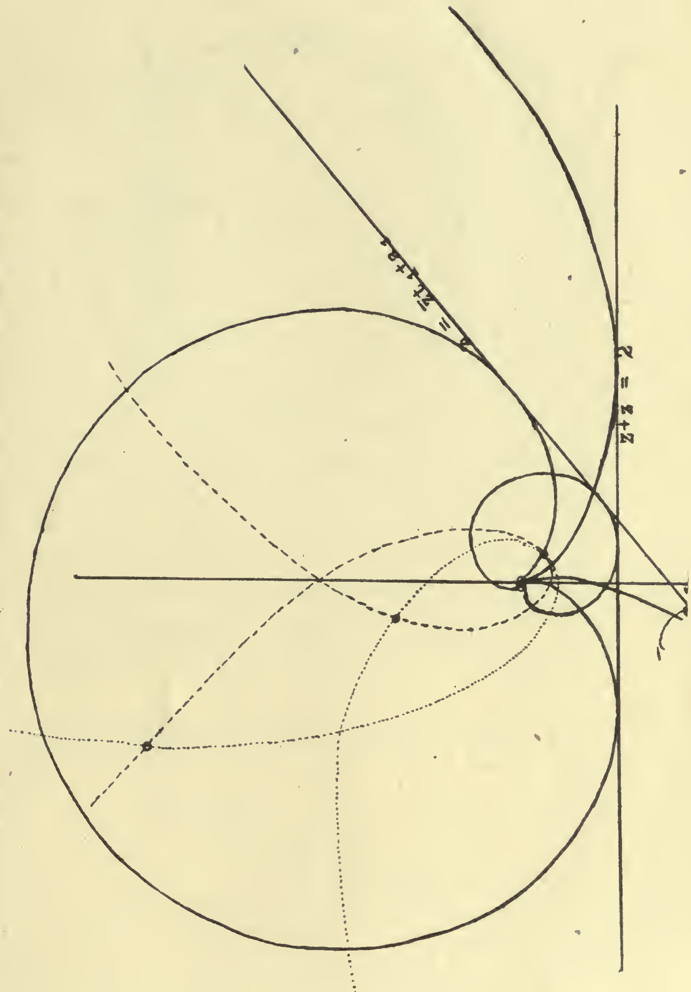

Fig. IX.—Showing the 3 real cardioids when the cusp and 2 lines are given.

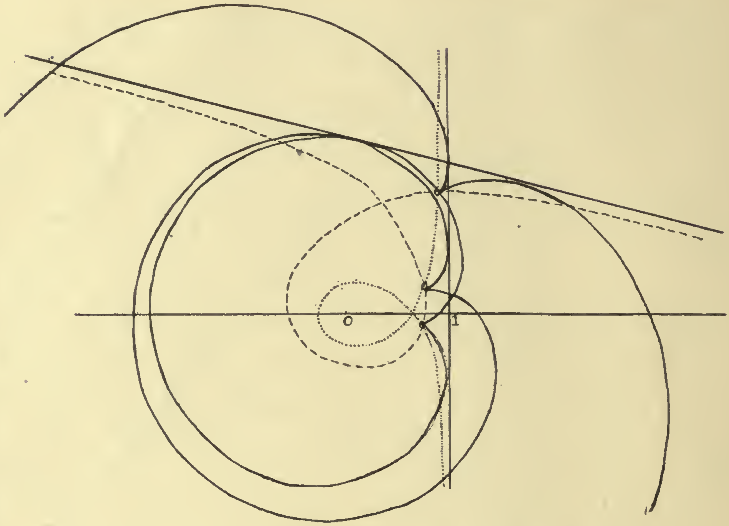

Fig. X.—Showing the 3 real cardioids when the centre and 2 lines are given.

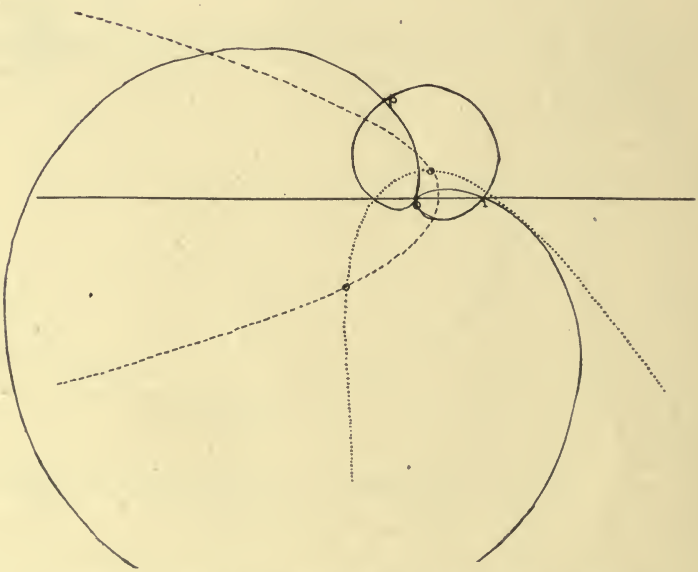

Fig. XI.—Showing the 2 real cardioids when the cusp and 2 points are given.

VITA

Sister Mary Gervase Kelley was born in Roxbury, Mass., September 8, 1888. She received her elementary education in St. Patrick's Parochial School and was graduated from the High School in 1905. In 1906 she entered the novitiate of the Sisters of Charity, Halifax, Nova Scotia, and there continued her studies in the Novitiate Normal School. From 1908 to 1913 she taught in the Halifax Public Schools. In 1910 she began work with the University of London, from which institution she received the Matriculation and the Intermediate Arts certificates. The four academic years since 1913, with the intervening Summer Sessions, have been spent in residence at the Catholic Sisters College, Catholic University of America, where she received the degree Bachelor of Arts in 1914, and that of Master of Arts in 1915. In her graduate work the principal courses followed have been those under Aubrey E. Landry, Ph.D., and John B. O'Connor, Ph.D., for the work done under both of whom it is the writer's pleasure to express her appreciation, and in particular to acknowledge gratefully the constant interest and kindly encouragement and assistance given by Dr. Landry, not only during the preparation of this dissertation, but during her entire University course.